Introduction to Robust Optimization

Part 2

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Outline

Conic Optimization

- A *conic* optimization (CO) problem (called also *conic program*) is of the form

$$\min_x \left\{ c^\top x + d : Ax - b \in K \right\}, \tag{1}$$

where $x \in \mathbb{R}^n$ is the decision vector, $K \subset \mathbb{R}^m$ is a closed pointed convex cone with a nonempty interior, and $x \mapsto Ax - b$ is a given affine mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$.

- an extremely wide variety of convex programs is covered by just three types of cones:

Conic Optimization

- **K** is a non-negative orthant $\mathbb{R}^m_+$. These cones give rise to Linear Optimization problems

$$\min_x \left\{ c^\top x : a_i^\top x - b_i \geq 0,\ 1 \leq i \leq m \right\}.$$

- **K** is a direct product of *Lorentz* (or *Second-order*) cones $\mathbf{L}^k = \{x \in \mathbb{R}^k : x_k \geq \sqrt{\sum_{j=1}^{k-1} x_j^2}\}$.
These cones give rise to Conic Quadratic Optimization (called also Second Order Conic Optimization). The Mathematical Programming form of a CQO problem is

$$\min_x \left\{ c^\top x : \|A_i x - b_i\|_2 \leq c_i^\top x - d_i,\ 1 \leq i \leq m \right\}.$$

- **K** is a direct product of *semidefinite* cones $\mathbf{S}^k_+$. The family of semidefinite cones gives rise to *Semidefinite Optimization* (SDO) – optimization programs of the form

$$\min_x \left\{ c^\top x + d : \mathcal{A}_i x - B_i \succeq 0,\ 1 \leq i \leq m \right\},$$

where

$$x \mapsto \mathcal{A}_i x - B_i \equiv \sum_{j=1}^{n} x_j A^{ij} - B_i$$

is an affine mapping from $\mathbb{R}^n$ to $\mathbf{S}^{k_i}$ (so that A^{ij} and B_i are symmetric $k_i \times k_i$ matrices), and $A \succeq 0$ means that A is a symmetric positive semidefinite matrix.

Robust Conic Problems

- Assume the *data* in the conic problem are uncertain. Specifically, we let $A = A(\zeta)$, $b = b(\zeta)$ be affine functions of $\zeta \in \mathcal{Z}$, where $\mathcal{Z}$ is a convex uncertainty set.
- Then, the robust version of the generic conic program is:

$$\min_x \left\{ c^\top x + d : A(\zeta)x - b(\zeta) \in K, \ \forall \zeta \in \mathcal{Z} \right\}. \tag{2}$$

- The objective is assumed to be certain, without loss of generality.

Robust Conic Problems

A general solvable case: polytopic uncertainty

- Assume $\mathcal{Z} = \mathrm{co}\{\zeta^{(1)}, \ldots, \zeta^{(N)}\}$.
- Then the robust conic problem (2) is solvable exactly as:

$$\min_x \left\{ c^\top x + d : A(\zeta^{(i)})x - b(\zeta^{(i)}) \in K,\ i = 1, \ldots, N \right\}.$$

Proof: Let x be feasible for the above problem, and write a generic point in $\mathcal{Z}$ as a convex combination $\zeta = \sum_{i=1}^N \theta_i \zeta^{(i)}$, $\theta_i \geq 0$, $\sum_{i=1}^N \theta_i = 1$. Recall $A(\zeta), b(\zeta)$ are affine in ζ:

$$(A(\zeta), b(\zeta)) = (A_0, b_0) + \sum_{\ell=1}^{L} \zeta_\ell (A_\ell, b_\ell).$$

Then,

$$A(\zeta)x - b(\zeta) = \sum_{i=1}^{N} \theta_i \left(A(\zeta^{(i)})x - b(\zeta^{(i)}) \right),$$

from which the statement immediately follows.

Next discuss further solvable cases, for specific cones (Lorentz cones, semidefinite cones).

Outline

Robust Second Order Cone (SOC) Optimization

- Consider robust SOC problems of the form:

$$\min_x \; c^\top x$$
$$\|A_i(\zeta_i)x - b_i(\zeta_i)\|_2 \le c_i(\zeta_i)^\top x - d_i(\zeta_i), \; \forall \zeta_i \in \mathcal{Z}_i, \; 1 \le i \le m$$

where $A_i(\zeta_i) \in \mathbb{R}^{k \times n}$, $b_i(\zeta_i) \in \mathbb{R}^k$, $c_i(\zeta_i) \in \mathbb{R}^n$, $d_i(\zeta_i) \in \mathbb{R}$ are affine in ζ_i.

- Without loss of generality, we may concentrate on a single constraint:

$$\| \underbrace{A(\zeta)x + b(\zeta)}_{\equiv \alpha(x)\zeta + \beta(x)} \|_2 \le \underbrace{c^\top(\zeta)x + d(\zeta)}_{\equiv \sigma^\top(x)\zeta + \delta(x)}, \; \forall \zeta \in \mathcal{Z}. \qquad (3)$$

where $\alpha(x)$, $\beta(x)$, $\sigma(x)$, $\delta(x)$ are affine in x.

Robust Second Order Cone (SOC) Optimization

- Assume $\zeta = (\eta, \xi)$, $\eta \in \mathcal{Z}^{\text{left}}$, $\xi \in \mathcal{Z}^{\text{right}}$, so that constraint (3) reads

$$\|A(\eta)x + b(\eta)\|_2 \leq c^\top(\xi)x + d(\xi), \ \forall \eta \in \mathcal{Z}^{\text{left}}, \xi \in \mathcal{Z}^{\text{right}}$$

- Assume the right hand side perturbation set is described by a conic representation:

$$\mathcal{Z}^{\text{right}} = \{\xi : \exists u : P\xi + Qu + p \in \mathbf{K}\},$$

  where $\mathbf{K}$ is a closed convex pointed cone.

- We next discuss two cases where the robust SOC constraint has an exact tractable representation. Namely, when $\mathcal{Z}^{\text{left}}$ is either an hyperrectangle (interval uncertainty) or a norm-bounded set.

Robust SOC Problems with Interval Uncertainty

- Assume $\mathcal{Z}^{\rm left}$ is an hyperrectangle:

$$\begin{aligned}
\mathcal{Z}^{\rm left} &= \{\eta = [\delta A, \delta b] : |(\delta A)_{ij}| \leq \delta_{ij}, \\
&\quad 1 \leq i \leq k, 1 \leq j \leq n, \\
&\quad |(\delta b)_i| \leq \delta_i, \ 1 \leq i \leq k\}, \\
[A(\zeta), b(\zeta)] &= [\bar{A}, \bar{b}] + [\delta A, \delta b].
\end{aligned}$$

- In other words, every entry in the left hand side data $[A, b]$, independently of all other entries, runs through a given segment centered at the nominal value of the entry.

Robust SOC Problems with Interval Uncertainty

Theorem

Under the previous hypotheses, the robust SOC constraint (3) is equivalent to the following explicit system of conic quadratic and linear constraints in variables x, z, τ, v:

$$
\begin{array}{|c|l|}
\hline
(a) & \begin{array}{l} \tau + p^\top v \le \delta(x), \;\; P^\top v = \sigma(x), \\ Q^\top v = 0, \;\; v \in \mathbf{K}_* \end{array} \\
\hline
(b) & \begin{array}{l} z_i \ge |(\bar{A}x + \bar{b})_i| + \delta_i + \sum_{j=1}^{n} |\delta_{ij} x_j|, \; i = 1, \ldots, k \\ \|z\|_2 \le \tau \end{array} \\
\hline
\end{array}
\qquad (4)
$$

where $\mathbf{K}_*$ is the cone dual to $\mathbf{K}$.

Robust SOC Problems with Interval Uncertainty

Sketch of proof

- Due to the side-wise structure of the uncertainty, a given x is robust feasible if and only if there exists τ such that

$$
\begin{aligned}
(a)\quad & \tau \le \min_{\xi\in\mathcal{Z}^{\text{right}}} \left\{\sigma^\top(y)\xi + \delta(x)\right\} \\
& = \min_{\xi,u} \left\{\sigma^\top(x)\xi : P\xi + Qu + p \in \mathbf{K}\right\} + \delta(x), \\
(b)\quad & \tau \ge \max_{\eta\in\mathcal{Z}^{\text{left}}} \|A(\eta)x + b(\eta)\|_2 \\
& = \max_{\delta A,\delta b} \left\{\|[\bar{A}x + \bar{b}] + [\delta A x + \delta b]\|_2 : |\delta A|_{ij} \le \delta_{ij}, |\delta b_i| \le \delta_i\right\}.
\end{aligned}
$$

- By ▸ Conic Duality, a given τ satisfies (a) if and only if τ can be extended, by properly chosen v, to a solution of $(4.a)$; by evident reasons, τ satisfies (b) if and only if there exists z satisfying $(4.b)$. □

Robust SOC Problems with Norm-Bounded Uncertainty

- Assume $\mathcal{Z}^{\text{left}}$ is a norm ball:

$$\mathcal{Z}^{\text{left}} = \{\eta \in \mathbb{R}^{p\times q} : \|\eta\|_{2,2} \leq 1\} \tag{5}$$

- And either

$$A(\eta)x + b(\eta) = \bar{A}x + \bar{b} + L^\top(x)\eta R \tag{6}$$

with $L(x)$ affine in x and $R \neq 0$, or

$$A(\eta)x + b(\eta) = \bar{A}x + \bar{b} + L^\top \eta R(x) \tag{7}$$

with $R(x)$ affine in x and $L \neq 0$.

- Here

$$\|\eta\|_{2,2} = \max_u \{\|\eta u\|_2 : u \in \mathbb{R}^q, \|u\|_2 \leq 1\}$$

is the usual matrix norm of a $p \times q$ matrix η (the maximal singular value).

Robust SOC Problems with Norm-Bounded Uncertainty

Theorem

Under the previous hypotheses, the robust SOC constraint (3) is equivalent to the following explicit system of LMIs in variables x, τ, u, λ:

(i) In the case of left hand side perturbations (5), (6):

$$\begin{array}{|c|c|}\hline (a) & \begin{array}{l}\tau + p^\top v \le \delta(x), \;\; P^\top v = \sigma(x), \\ Q^\top v = 0, \;\; v \in \mathbf{K}_* \end{array} \\ \hline (b) & \left[\begin{array}{c|c|c} \tau I_k & r^\top L^\top(x) & \widehat{x} \\ \hline L(x) r & \lambda I_{n+1} & \\ \hline \widehat{x}^\top & & \tau - \lambda R^\top R \end{array}\right] \succeq 0. \\ \hline \end{array} \tag{8}$$

(ii) In the case of left hand side perturbations (5), (7):

$$\begin{array}{|c|c|}\hline (a) & \begin{array}{l}\tau + p^\top v \le \delta(x), \;\; P^\top v = \sigma(x), \\ Q^\top v = 0, \;\; v \in \mathbf{K}_* \end{array} \\ \hline (b) & \left[\begin{array}{c|c|c} \tau I_k - \lambda L^\top L & & \widehat{x} \\ \hline & \lambda I_q & R(x) \\ \hline \widehat{x}^\top & R^\top(x) & \tau \end{array}\right] \succeq 0. \\ \hline \end{array} \tag{9}$$

Here, $\mathbf{K}_*$ is the cone dual to $\mathbf{K}$, and $\widehat{x} = \bar{A}x + \bar{b}$.

Robust SOC Problems with Norm-Bounded Uncertainty

The proof of the previous theorem (skipped) is based on three key facts:

1. **Schur Complement Lemma**
   Let $R \succ 0$. Then,
   $$A = \left[\begin{array}{c|c} P & Q^\top \\ \hline Q & R \end{array}\right] \succeq 0$$
   if and only if $P - Q^\top R^{-1} Q \succeq 0$.

2. **Semidefinite Representation of Lorentz cone**
   A vector $[y; t] \in \mathbb{R}^k \times \mathbb{R}$ belongs to the Lorentz cone $\mathbf{L}^{k+1} = \{[y; t] \in \mathbb{R}^{k+1} : t \geq \|y\|_2\}$ if and only if the "arrow matrix"
   $$\mathrm{Arrow}(y, t) = \left[\begin{array}{c|c} t & y^\top \\ \hline y & tI_k \end{array}\right]$$
   is positive semidefinite.

3. **$\mathcal{S}$ Lemma**
   (i) [homogeneous version] Let A, B be symmetric matrices of the same size such that $\bar{x}^\top A\bar{x} > 0$ for some $\bar{x}$. Then the implication
   $$x^\top Ax \geq 0 \Rightarrow x^\top Bx \geq 0$$
   holds true if and only if $\exists \lambda \geq 0 : B \succeq \lambda A$.
   (ii) [inhomogeneous version] Let A, B be symmetric matrices of the same size, and let the quadratic form $x^\top Ax + 2a^\top x + \alpha$ be strictly positive at some point. Then the implication
   $$x^\top Ax + 2a^\top x + \alpha \geq 0 \Rightarrow x^\top Bx + 2b^\top x + \beta \geq 0$$
   holds true if and only if $\exists \lambda \geq 0 : \left[\begin{array}{c|c} B - \lambda A & b^\top - \lambda a^\top \\ \hline b - \lambda a & \beta - \lambda\alpha \end{array}\right] \succeq 0.$

Outline

Robust Semidefinite Optimization

- A *semidefinite program* (SDP) is a conic optimization program

$$\min_x \left\{ c^\top x + d : \mathcal{A}_i(x) \equiv \sum_{j=1}^{n} x_j A^{ij} - B_i \in \mathbf{S}_+^{k_i},\ i = 1, \ldots, m \right\}$$
$$\Updownarrow$$
$$\min_x \left\{ c^\top x + d : \mathcal{A}_i(x) \equiv \sum_{j=1}^{n} x_j A^{ij} - B_i \succeq 0,\ i = 1, \ldots, m \right\}$$

where A^{ij}, B_i are symmetric matrices of sizes $k_i \times k_i$, $\mathbf{S}_+^k$ is the cone of real symmetric positive semidefinite $k \times k$ matrices, and $A \succeq B$ means that A, B are symmetric matrices of the same sizes such that the matrix $A - B$ is positive semidefinite.

- A constraint of the form

$$\mathcal{A}x - B \equiv \sum_j x_j A^j - B \succeq 0$$

with symmetric A^j, B is called a *Linear Matrix Inequality* (LMI); thus, an SDP is the problem of minimizing a linear objective under finitely many LMI constraints.

Robust Semidefinite Optimization

A robust SDP is an optimization problem of the form

$$\min_x \left\{ c^\top x + d : \mathcal{A}_\zeta(x) \succeq 0,\ \forall \zeta \in \rho\mathcal{Z} \right\},$$

where

$$\mathcal{A}_\zeta(x) \equiv \bar{\mathcal{A}}(x) + \sum_{\ell=1}^{L} \zeta_\ell \mathcal{A}_\ell(x)$$

and where

- $\bar{\mathcal{A}}(x)$, $\mathcal{A}_\ell(x)$ are symmetric matrices affinely depending on the design vector x
- $\mathcal{Z}$ is the uncertainty set
- $\rho \geq 0$ is the uncertainty level.

Robust Semidefinite Optimization

Exact tractable representations of robust SDPs are known only in two special cases:

- Polytopic uncertainty: $\mathcal{Z} = \mathrm{co}\{\zeta^{(1)}, \ldots, \zeta^{(\ell)}\}$ (this is true for general robust conic problems);
- Norm-bounded unstructured uncertainty:

$$\mathcal{Z} = \{\zeta : \|\zeta\|_{2,2} \leq 1\}.$$

The latter case is discussed next in detail.

Robust Semidefinite Problems with Norm-Bounded Uncertainty

- We consider a robust SDP

$$\min_x \left\{ c^\top x + d : \mathcal{A}_\zeta(x) \succeq 0, \ \forall \zeta \in \rho\mathcal{Z} \right\},$$

- We assume that

$$\mathcal{A}_\zeta(x) = \bar{A}(x) + \left[L^\top(x)\zeta R + R^\top \zeta^\top L(x) \right],$$

where $L(\cdot)$ is affine in x;

- The perturbation set $\mathcal{Z}$ is the set of all $p \times q$ matrices ζ with the usual matrix norm $\|\cdot\|_{2,2}$ not exceeding 1.

Robust Semidefinite Problems with Norm-Bounded Uncertainty

We have that:

$$
\begin{array}{rc}
 & x \text{ is robust feasible at uncertainty level } \rho \\
\Leftrightarrow & \xi^\top[\bar{A}(x) + L^\top(x)\zeta R + R^\top \zeta^\top L(x)]\xi \geq 0 \quad \forall \xi \quad \forall (\zeta : \|\zeta\|_{2,2} \leq \rho) \\
\Leftrightarrow & \xi^\top \bar{A}(x)\xi + 2\xi^\top L^\top(x)\zeta R\xi \geq 0 \quad \forall \xi \quad \forall (\zeta : \|\zeta\|_{2,2} \leq \rho) \\
\Leftrightarrow & \xi^\top \bar{A}(x)\xi + 2 \underbrace{\min_{\|\zeta\|_{2,2}\leq\rho} \xi^\top L^\top(x)\zeta R\xi}_{=-\rho\|L(x)\xi\|_2\|R\xi\|_2} \geq 0 \quad \forall \xi \\
\Leftrightarrow & \xi^\top \bar{A}(x)\xi - 2\rho\|L(x)\xi\|_2\|R\xi\|_2 \geq 0 \ \forall \xi \\
\Leftrightarrow & \xi^\top \bar{A}(x)\xi + 2\rho\eta^\top L(x)\xi \geq 0 \quad \forall (\xi, \eta : \eta^\top \eta \leq \xi^\top R^\top R\xi) \\
\Leftrightarrow & \exists \lambda \geq 0 : \left[\begin{array}{c|c} & \rho L(x) \\ \hline \rho L^\top(x) & \bar{A}(x) \end{array}\right] \succeq \lambda \left[\begin{array}{c|c} -I_p & \\ \hline & R^\top R \end{array}\right] \\
 & [\mathcal{S}\text{-Lemma}] \\
\Leftrightarrow & \exists \lambda : \left[\begin{array}{c|c} \lambda I_p & \rho L(x) \\ \hline \rho L^\top(x) & \bar{A}(x) - \lambda R^\top R \end{array}\right] \succeq 0.
\end{array}
$$

Theorem

The robust LMI constraint

$$\mathcal{A}^{\mathrm{n}}(x) + L^\top(x)\zeta R + R^\top \zeta^\top L(x) \succeq 0 \quad \forall (\zeta \in \mathbb{R}^{p\times q} : \|\zeta\|_{2,2} \leq \rho)$$

can be represented equivalently by the LMI

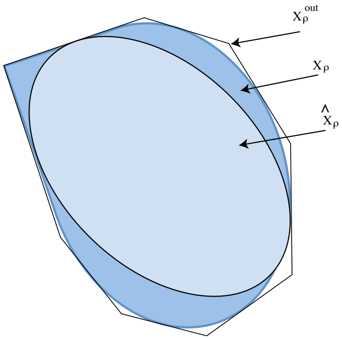
$$\left[\begin{array}{c|c} \lambda I_p & \rho L(x) \\ \hline \rho L^\top(x) & \bar{\mathcal{A}}(x) - \lambda R^\top R \end{array}\right] \succeq 0 \tag{10}$$

in variables x, λ.

Outline

Robust Conic Optimization: Tight tractable approximations

- We have seen that robust conic programs can be represented in a computationally tractable form only in some very special cases (for example, in the semidefinite case, only for polytopic and unstructured norm-bounded uncertainty);
- For more general uncertainty structures we need to resort to *approximations*;
- Approximation approaches aim at approximating the robust feasible set X either from outside (risky) or from inside (safe);
- Here we discuss tight safe approximations of robust conic problems.



Robust Conic Optimization: Tight tractable approximations

- Consider a robust conic constraint (RC)

$$\underbrace{A(\zeta)x + b(\zeta)}_{\equiv \alpha(x)\zeta + \beta(x)} \in \mathbf{Q} \ \forall \zeta \in \mathcal{Z}, \tag{11}$$

where $A(\zeta) \in \mathbb{R}^{k \times n}$, $b(\zeta) \in \mathbb{R}^k$ are affine in ζ, so that $\alpha(x)$, $\beta(x)$ are affine in x.

- We say that a system $\mathcal{S}$ of convex constraints in variables x and, perhaps, additional variables u is a safe approximation of the RC (11), if the projection of the feasible set of $\mathcal{S}$ on the space of x-variables is contained in the feasible set of the RC:

$$\forall x : (\exists u : \ (x, u) \text{ satisfies } \mathcal{S}) \Rightarrow x \text{ satisfies } (11).$$

- This approximation is called tractable, provided that $\mathcal{S}$ is so (e.g., $\mathcal{S}$ is an explicit system of CQI's/LMIs or, more generally, the constraints in $\mathcal{S}$ are efficiently computable.

Quantifying the conservatism

- Assume that $0 \in \mathcal{Z}$, and consider a single-parametric family of perturbation sets

$$\mathcal{Z}_\rho = \rho \mathcal{Z}, \; 0 < \rho \le \infty, \tag{12}$$

thus giving rise to a single-parametric family

$$\underbrace{A(\zeta)x + b(\zeta)}_{\equiv \alpha(x)\zeta + \beta(x)} \in \mathbf{Q} \; \forall \zeta \in \mathcal{Z}_\rho \tag{RC$_\rho$}$$

of robust conic constraints.

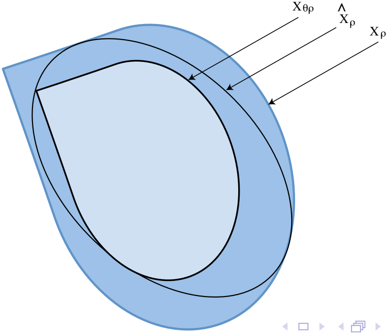
- ρ is the *perturbation level*; the original perturbation set $\mathcal{Z}$ corresponds to the perturbation level $\rho = 1$.
- The feasible set X_ρ shrinks as ρ grows.

Robust Conic Optimization: Tight tractable approximations

Quantifying the conservatism

- Assume that we are given an approximation scheme which puts into correspondence to $\mathcal{Z}_\rho$ a finite system $\mathcal{S}_\rho$ of efficiently computable convex constraints on variables x and, perhaps, additional variables u, depending on $\rho > 0$, in such a way that for every ρ the system $\mathcal{S}_\rho$ is a safe tractable approximation of (RC$_\rho$), and let $\widehat{X}_\rho$ be the projection of the feasible set of $\mathcal{S}_\rho$ onto the space of x-variables.
- The conservatism (or "tightness factor") of the approximation scheme in question does not exceed $\vartheta \geq 1$, if, for every $\rho > 0$, we have

$$X_{\vartheta\rho} \subset \widehat{X}_\rho \subset X_\rho .$$



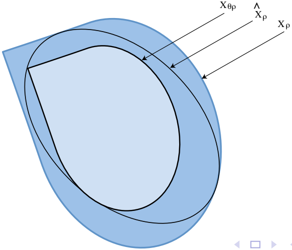
Robust Conic Optimization: Tight tractable approximations

Quantifying the conservatism

The fact that $\mathcal{S}_\rho$ is a safe approximation of (RC$_\rho$) tight within factor ϑ is equivalent to the following pair of statements:

1. [safety] *Whenever a vector x and $\rho > 0$ are such that x can be extended to a feasible solution of $\mathcal{S}_\rho$, x is feasible for* (RC$_\rho$);
2. [tightness] *Whenever a vector x and $\rho > 0$ are such that x cannot be extended to a feasible solution of $\mathcal{S}_\rho$, x is not feasible for* (RC$_{\vartheta\rho}$).

Clearly, tightness factor equal to 1 means that the approximation is precise: $\widehat{X}_\rho = X_\rho$ for all ρ. In many applications, especially in those where the level of perturbations is known only "up to an order of magnitude", a safe approximation of the RC with moderate tightness factor is almost as useful, from a practical viewpoint, as the RC itself.



Tight Tractable Approximations of Robust SDP

- The possibility to reformulate a robust semidefinite program in a computationally tractable form is a “rare commodity”. Essentially, only the polytopic case and the unstructured norm-bounded case can be reformulated exactly.
- For other cases, we are thus interested in tight tractable *approximations*.
- We next discuss a quite general case of *structured norm-bounded uncertainty*.

Preliminaries: The Matrix Cube Theorem

- *Let m, p_1, q_1,...,p_L, q_L be positive integers, and $A \in \mathbf{S}^m$, $L_j \in \mathbb{R}^{p_j \times m}$, $R_j \in \mathbb{R}^{q_j \times m}$ be given matrices, $L_j \neq 0$. Let also a partition $\{1, 2, \ldots, L\} = I_s^r \cup I_f^r$ of the index set $\{1, \ldots, L\}$ into two non-overlapping sets be given.*
- *Associate a parametric family of "matrix boxes"*

$$\mathcal{U}[\rho] = \left\{ A + \rho \sum_{j=1}^{L} [L_j^\top \Theta^j R_j + R_j^\top [\Theta^j]^\top L_j] : \Theta^j \in \mathcal{Z}^j, 1 \leq j \leq L \right\} \subset \mathbf{S}^m, \tag{13}$$

where $\rho \geq 0$ is the parameter and

$$\mathcal{Z}^j = \begin{cases} \{\theta I_{p_j} : \theta \in \mathbb{R}, |\theta| \leq 1\}, j \in I_s^r \\ \quad \text{[``scalar perturbation blocks'']} \\ \{\Theta^j \in \mathbb{R}^{p_j \times q_j} : \|\Theta^j\|_{2,2} \leq 1\}, j \in I_f^r \\ \quad \text{[``full perturbation blocks'']} \end{cases} . \tag{14}$$

- *Matrix Cube Problem: Given $\rho \geq 0$, check whether*

$$\mathcal{U}[\rho] \subset \mathbf{S}_+^m \qquad \mathcal{A}(\rho)$$

- Consider, along with predicate $\mathcal{A}(\rho)$, the predicate

$$\begin{array}{ll} \exists Y_j \in \mathbf{S}^m, \ j = 1, \ldots, L : & \\ (a) \quad Y_j \succeq L_j^\top \Theta^j R_j + R_j^\top [\Theta^j]^\top L_j \ \forall \left(\Theta^j \in \mathcal{Z}^j, 1 \leq j \leq L\right) & \\ (b) \quad A - \rho \sum_{j=1}^{L} Y_j \succeq 0. & \end{array} \qquad \mathcal{B}(\rho)$$

The Matrix Cube Theorem

- Predicate $\mathcal{B}(\rho)$ is stronger than $\mathcal{A}(\rho)$ – the validity of the former predicate implies the latter.
- $\mathcal{B}(\rho)$ is computationally tractable – the validity of the predicate is equivalent to the solvability of the system of LMIs

$$\begin{array}{lrcl} (s) & Y_j \pm \left[L_j^\top R_j + R_j^\top L_j\right] & \succeq & 0,\ j \in I_{\mathrm{s}}^{\mathrm{r}}, \\ (f) & \begin{bmatrix} Y_j - \lambda_j L_j^\top L_j & R_j^\top \\ R_j & \lambda_j I_{p_j} \end{bmatrix} & \succeq & 0,\ j \in I_{\mathrm{f}}^{\mathrm{r}} \\ (*) & A - \rho \sum_{j=1}^{L} Y_j & \succeq & 0. \end{array} \tag{15}$$

in matrix variables $Y_j \in \mathbf{S}^m$, $j = 1, \ldots, L$, and real variables λ_j, $j \in I_{\mathrm{f}}^{\mathrm{r}}$. aaaa
- "The gap" between $\mathcal{A}(\rho)$ and $\mathcal{B}(\rho)$ can be bounded solely in terms of the maximal rank

$$p^{\mathrm{s}} = \max_{j \in I_{\mathrm{s}}^{\mathrm{r}}} \mathrm{rank}(L_j^\top R_j + R_j^\top L_j)$$

of the scalar perturbations. Specifically, there exists a universal function $\vartheta_{\mathbb{R}}(\cdot)$ satisfying the relations

$$\vartheta_{\mathbb{R}}(2) = \frac{\pi}{2};\ \vartheta_{\mathbb{R}}(4) = 2;\ \vartheta_{\mathbb{R}}(\mu) \leq \pi\sqrt{\mu}/2\ \forall \mu \geq 1$$

such that with $\mu = \max[2, p^{\mathrm{s}}]$ one has

$$\text{if } \mathcal{B}(\rho) \text{ is not valid, then } \mathcal{A}(\vartheta_{\mathbb{R}}(\mu)\rho) \text{ is not valid.} \tag{16}$$

- Finally, in the case $L = 1$ of single perturbation block $\mathcal{A}(\rho)$ is equivalent to $\mathcal{B}(\rho)$.

Robust LMI with structured norm-bounded uncertainty

- Consider an uncertain LMI
$$\mathcal{A}_\zeta(y) \succeq 0 \tag{17}$$
where the "body" $\mathcal{A}_\zeta(y)$ is bi-linear in the design vector y and the perturbation vector ζ.
- We say that the uncertain constraint (17) is affected by structured norm-bounded uncertainty with uncertainty level ρ, if
1. The perturbation set $\mathcal{Z}_\rho$ is of the form

$$\mathcal{Z}_\rho = \left\{ \zeta = (\zeta^1, \dots, \zeta^L) : \begin{array}{l} \zeta^\ell \in \mathbb{R}, |\zeta^\ell| \leq \rho, \ \ell \in \mathcal{I}_s \\ \zeta^\ell \in \mathbb{R}^{p_\ell \times q_\ell} : \|\zeta^\ell\|_{2,2} \leq \rho, \ \ell \notin \mathcal{I}_s \end{array} \right\} \tag{18}$$

2. The body $\mathcal{A}_\zeta(y)$ of the constraint can be represented as

$$\begin{aligned} \mathcal{A}_\zeta(y) &= \mathcal{A}^{\mathrm{n}}(y) + \sum_{\ell \in \mathcal{I}_S} \zeta^\ell \mathcal{A}_\ell(y) \\ &\quad + \sum_{\ell \notin \mathcal{I}_s} \left[L_\ell^\top(y) \zeta^\ell R_\ell + R_\ell^\top [\zeta^\ell]^\top L_\ell(y) \right], \end{aligned} \tag{19}$$

where $\mathcal{A}_\ell(y)$, $\ell \in \mathcal{I}_s$, and $L_\ell(y)$, $\ell \notin \mathcal{I}_s$, are affine in y, and R_ℓ, $\ell \notin \mathcal{I}_s$, are nonzero.

Robust LMI with structured norm-bounded uncertainty

Theorem

- Given an uncertain LMI with structured norm-bounded uncertainty (18), (19), let us associate with it the following system of LMIs in variables Y_ℓ, $\ell = 1, \ldots, L$, λ_ℓ, $\ell \notin \mathcal{I}_s$, y:

$$
\begin{array}{ll}
(a) & Y_\ell \succeq \pm \mathcal{A}_\ell(y),\ \ell \in \mathcal{I}_s \\
(b) & \left[\begin{array}{c|c} \lambda_\ell I_{p_\ell} & L_\ell(y) \\ \hline L_\ell^\top(y) & Y_\ell - \lambda_\ell R_\ell^\top R_\ell \end{array}\right] \succeq 0,\ \ell \notin \mathcal{I}_s \\
(c) & \mathcal{A}^{\mathrm{n}}(y) - \rho \sum_{\ell=1}^{L} Y_\ell \succeq 0
\end{array} \tag{20}
$$

- Then system (20) is a safe tractable approximation of the robust constraint

$$
\mathcal{A}_\zeta(y) \succeq 0 \ \forall \zeta \in \mathcal{Z}_\rho \tag{21}
$$

and the tightness factor of this approximation does not exceed $\vartheta(\mu)$, where μ is the smallest integer ≥ 2 such that $\mu \geq \max_y \mathrm{rank}(\mathcal{A}_\ell(y))$ for all $\ell \in \mathcal{I}_s$, and $\vartheta(\cdot)$ is a universal function of μ such that

$$
\vartheta(2) = \frac{\pi}{2},\ \vartheta(4) = 2,\ \vartheta(\mu) \leq \pi\sqrt{\mu/2},\ \mu > 2.
$$

- The approximation is exact, if either $L = 1$, or all perturbations are scalar (i.e., $\mathcal{I}_s = \{1, \ldots, L\}$) and all $\mathcal{A}_\ell(y)$ are of ranks not exceeding 1.

Robust LMI with structured norm-bounded uncertainty

Proof

- Let us fix y and observe that a collection $y, Y_1, \ldots, Y_L$ can be extended to a feasible solution of (20) if and only if
$$\forall \zeta \in \mathcal{Z}_\rho : \left\{ \begin{array}{l} -\rho Y_\ell \preceq \zeta^\ell \mathcal{A}_\ell(y),\ \ell \in \mathcal{I}_\mathrm{s}, \\ -\rho Y_\ell \preceq L_\ell^\top(y)\zeta^\ell R_\ell + R_\ell^\top [\zeta^\ell]^\top L_\ell(y),\ \ell \notin \mathcal{I}_\mathrm{s} \end{array} \right.$$
(see the unstructured case).
- It follows that if, in addition, Y_ℓ satisfy (20.c), then y is feasible for (21), so that (20) is a safe tractable approximation of (21).
- The fact that this approximation is tight within the factor $\vartheta(\mu)$ is readily given by the Real case Matrix Cube Theorem.
- The fact that the approximation is exact when $L = 1$ is evident when $\mathcal{I}_\mathrm{s} = \{1\}$ and is readily given by the unstructured result when $\mathcal{I}_\mathrm{s} = \emptyset$.
- The fact that the approximation is exact when all perturbations are scalar and all matrices $\mathcal{A}_\ell(y)$ are of ranks not exceeding 1 is evident. □

Outline

Example: Lyapunov Stability Analysis

- Consider a time-varying linear dynamical system “closed” by a linear output-based feedback:

$$
\begin{array}{lrcl}
(a) & \dot{x}(t) & = & A_t x(t) + B_t u(t) + R_t d_t \text{ [open loop system, or } \textit{plant}] \\
(b) & y(t) & = & C_t x(t) + D_t d_t \text{ [output]} \\
(c) & u(t) & = & K_t y(t) \text{ [output-based feedback]} \\
 & & \Downarrow & \\
(d) & \dot{x}(t) & = & [A_t + B_t K_t C_t] x(t) + [R_t + B_t K_t D_t] d_t \text{ [closed loop system]}
\end{array}
\quad (22)
$$

- $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $d_t \in \mathbb{R}^p$, $y(t) \in \mathbb{R}^q$ are, respectively, the state, the control, the external disturbance and the output at time t, A_t, B_t, R_t, C_t, D_t are matrices of appropriate sizes specifying the dynamics of the system, and K_t is the feedback matrix.
- We assume that the dynamical system in question is *uncertain*, meaning that we do not know the dependencies of the matrices A_t,...,K_t on t; all we know is that the collection $M_t = (A_t, B_t, C_t, D_t, R_t, K_t)$ of all these matrices stays all the time within a given compact uncertainty set $\mathcal{M}$.
- We let A^{n},...,K^{n} represent an underlying time-invariant “nominal” system, while the actual dynamics corresponds to the case when the matrices drift (perhaps, in a time-dependent fashion) around their nominal values.

Example: Lyapunov Stability Analysis

- An important desired property of a linear dynamical system is its *stability* – the fact that every state trajectory $x(t)$ of (every realization of) the closed loop system converges to 0 as $t \to \infty$, provided that the external disturbances d_t are identically zero.
- For a time-invariant linear system
$$\dot{x} = Q^{\mathrm{n}} x,$$
the necessary and sufficient stability condition is that all eigenvalues of A have negative real parts or, equivalently, that there exists a *Lyapunov Stability Certificate* (LSC) – a positive definite symmetric matrix X such that
$$[Q^{\mathrm{n}}]^\top X + XQ^{\mathrm{n}} \prec 0.$$
- For uncertain system (22), a *sufficient* stability condition is that all matrices
$$Q \in \mathcal{Q} = \{Q = A^M + B^M K^M C^M : M \in \mathcal{M}\}$$
have a common LSC X, that is, there exists $X \succ 0$ such that
$$\begin{array}{ll} (a) & Q^\top X + XQ^\top \prec 0 \quad \forall Q \in \mathcal{Q} \\ & \Updownarrow \\ (b) & [A^M + B^M K^M C^M]^\top X + X[A^M + B^M K^M C^M] \prec 0 \quad \forall M \in \mathcal{M}; \end{array} \tag{23}$$
here A^M,...,K^M are the components of a collection $M \in \mathcal{M}$.

Example: Lyapunov Stability Analysis

- The set $\mathcal{Q}$ is compact along with $\mathcal{M}$. It follows that X is a LSC if and only if $X \succ 0$ and

$$\begin{array}{rl} & \exists \beta > 0 : Q^\top X + XQ \preceq -\beta I \quad \forall Q \in \mathcal{Q} \\ \Leftrightarrow & \exists \beta > 0 : Q^\top X + XQ \preceq -\beta I \quad \forall Q \in \mathrm{co}(\mathcal{Q}). \end{array}$$

- Multiplying such an X by an appropriate positive real, we can ensure that

$$X \succeq I \;\&\; Q^\top X + XQ \preceq -I \quad \forall Q \in \mathrm{co}(\mathcal{Q}). \tag{24}$$

  Thus, we lose nothing when requiring from LSC to satisfy the latter system of (semi-infinite) LMIs.
- Observe that (24) is nothing but the RC of the uncertain system of LMIs

$$X \succeq I \;\&\; Q^\top X + XQ \preceq -I, \tag{25}$$

  the uncertain data being Q and the uncertainty set being $\mathrm{co}(\mathcal{Q})$. Thus, RC's arise naturally in the context of Robust Control.
- We next study the question of existence of a LSC in two cases: *polytopic* and *unstructured norm-bounded* uncertainty.

Example: Lyapunov Stability Analysis

Polytopic uncertainty

- Polytopic uncertainty means that the set $\mathrm{co}(\mathcal{Q})$ is given as a convex hull of an explicit list of "scenarios" Q^i, $i = 1, \ldots, N$:

$$\mathrm{co}(\mathcal{Q}) = \mathrm{co}\{Q^1, \ldots, Q^N\}.$$

- This situation occurs when the components A^M, B^M, C^M, K^M of $M \in \mathcal{M}$ run, independently of each other, through convex hulls of respective scenarios

$$S_A = \mathrm{co}\{A^1, \ldots, A^{N_A}\}, S_B = \mathrm{co}\{B^1, \ldots, B^{N_B}\},$$
$$S_C = \mathrm{co}\{C^1, \ldots, C^{N_C}\}, S_K = \mathrm{co}\{K^1, \ldots, K^{N_K}\};$$

- In this case, the set $\mathrm{co}(\mathcal{Q})$ is nothing but the convex hull of $N = N_A N_B N_C N_K$ "scenarios" $Q^{ijk\ell} = A^i + B^j K^\ell C^k$, $1 \leq i \leq N_A$,...,$1 \leq \ell \leq N_K$.
- In the case in question we are in the situation of scenario perturbations, so that (25) is equivalent to the explicit system of LMIs

$$X \succeq I, [Q^i]^\top X + XQ^i \preceq -I, \; i = 1, \ldots, N.$$

Example: Lyapunov Stability Analysis

Unstructured norm-bounded uncertainty

- Here
$$\mathrm{co}(\mathcal{Q}) = \{Q = Q^{\mathrm{n}} + U\zeta V : \zeta \in \mathbb{R}^{p\times q},\ \|\zeta\|_{2,2} \leq \rho\}.$$
- In our context this situation occurs, e.g., when 3 of the four matrices $A^M, B^M, C^M, K^M, M \in \mathcal{M}$, are in fact certain, and the remaining matrix, say, A^M, runs through a set of the form $\{A^{\mathrm{n}} + G\zeta H : \zeta \in \mathbb{R}^{p\times q},\ \|\zeta\|_{2,2} \leq \rho\}$.
- In the case of unstructured norm-bounded uncertainty, the semi-infinite LMI in (25) is of the form

$$Q^\top X + XQ \preceq -I \quad \forall Q \in \mathrm{co}(\mathcal{Q})$$
$$\Updownarrow$$
$$\underbrace{-I - [Q^{\mathrm{n}}]^\top X - XQ^{\mathrm{n}}}_{\mathcal{A}^{\mathrm{n}}(X)} + [\underbrace{-XU}_{L^\top(X)}\, \zeta \underbrace{V}_{R} + V^\top \zeta^\top L(X)] \succeq 0$$
$$\forall(\zeta \in \mathbb{R}^{p\times q},\ \|\zeta\|_{2,2} \leq \rho)$$

- The robust LMI (25) is equivalent to the explicit system of LMIs

$$X \succeq I, \left[\begin{array}{c|c} \lambda I_p & -\rho U^\top X \\ \hline -\rho XU & -I - [Q^{\mathrm{n}}]^\top X - XQ^{\mathrm{n}} - \lambda V^\top V \end{array}\right] \succeq 0 \qquad (26)$$

in variables X, λ.

Lyapunov Stability Synthesis

- A more challenging problem is *Stability Synthesis*: given an uncertain open loop system (22.*a*-*b*) along with the associated compact uncertainty set $\widehat{\mathcal{M}}$ in the space of collections $\widehat{M} = (A, B, C, D, R)$, find a linear output-based feedback

$$u(t) = Ky(t)$$

and a LSC for the resulting closed loop system.

- The Synthesis problem has a nice solution in the state-feedback case (that is, $C_t \equiv I$), so that the state dynamics of the closed loop system is given by

$$\dot{x}(t) = [A_t + B_t K]x(t) + [R_t + B_t K D_t]d_t. \tag{27}$$

- The pairs (K, X) of "feedback–LSC" we are looking for are exactly the feasible solutions to the system of semi-infinite matrix inequalities in variables X, K:

$$X \succ 0 \;\&\; [A + BK]^\top X + X[A + BK] \prec 0 \;\; \forall [A, B] \in \mathcal{AB}; \tag{28}$$

here $\mathcal{AB}$ is the projection of $\widehat{M}$ on the space of $[A, B]$-data.

Lyapunov Stability Synthesis

- The difficulty is that the system is *nonlinear* in the variables. As a remedy, let us carry out the nonlinear substitution of variables $X = Y^{-1}$, $K = ZY^{-1}$. With this substitution, (28) becomes a system in the new variables Y, Z:

$$Y \succ 0 \;\&\; [A + BZY^{-1}]^\top Y^{-1} + Y^{-1}[A + BZY^{-1}] \prec 0 \;\; \forall [A, B] \in \mathcal{AB};$$

- Multiplying both sides of the second matrix inequality from the left and from the right by Y, we convert the system to the equivalent form

$$Y \succ 0, \;\&\; AY + YA^\top + BZ + Z^\top B^\top \prec 0 \quad \forall [A, B] \in \mathcal{AB};$$

- Since $\mathcal{AB}$ is compact along with $\widehat{\mathcal{M}}$, the solutions to the latter system are exactly the pairs (Y, Z) which can be obtained by scaling $(Y, Z) \mapsto (\lambda Y, \lambda Z)$, $\lambda > 0$, from the solutions to the system of semi-infinite LMIs

$$Y \succeq I \;\&\; AY + YA^\top + BZ + Z^\top B^\top \preceq -I \quad \forall [A, B] \in \mathcal{AB}. \tag{29}$$

in variables Y, Z.

- When the uncertainty $\mathcal{AB}$ can be represented either as a polytopic, or as an unstructured norm-bounded one, the system (29) of semi-infinite LMIs admits an equivalent tractable reformulation.

Outline

Conclusions

- Robust Optimization offers a series of techniques for dealing with *uncertainty* in the data of an optimization problem;
- For some simple uncertainty structures, robust conic problems can be converted exactly into standard conic problems (in lifted space), and therefore solved efficiently;
- More complex uncertainty structures (such as structured norm-bounded uncertainty) can be dealt with via safe (inner) approximations of the feasible set:
- General uncertainty can be dealt with via *risky* approximations, such as sample-based scenario counterparts.

That's all

Thanks for your attention!

Recall: Conic duality

- **Primal conic problem:**

$$\mathrm{Opt}(P) = \min_x \left\{ \langle c, x \rangle_E : \begin{array}{l} A_i x - b_i \in K_i, \; i = 1, \ldots, m, \\ Ax = b \end{array} \right\} \quad (P)$$

- **Dual problem:**

$$\mathrm{Opt}(D) = \max_{z, \{y_i\}} \left\{ \langle z, b \rangle_F + \sum_i \langle y_i, b_i \rangle_{F_i} : \begin{array}{l} y_i \in K_i^*, \; 1 \le i \le m, \\ A^* z + \sum_i A_i^* y_i = c \end{array} \right\} \quad (D)$$